International Journal of Theoretical Physics, Vol. 27, No. 5, 1988

Static Vacuum Solution of Direct Poincaré Gauge Theory in Ten Dimensions with Four External

Dominic G. B. Edelen[1]

Received December 16, 1987

A torsion-free solution of the free gauge field equations of direct Poincaré gauge theory on a ten-dimensional Minkowski space is constructed. This solution exhibits nontrivial curvature two-forms, but shaves the metric structure down to that of a four-dimensional Minkowski space. Universality of this solution with respect to the choice of the free field Lagrangian is established.

Current research into the foundations of elementary physical processes has led to a sizable investment in theories where the dimension of the underlying space is greater than four (supersymmetric theories in ten dimensions, string theory in 26 dimensions, etc.). Ultimate validation of such theories clearly requires a mechanism or procedure for reconciling these higher dimensional constructs with the unavoidable fact that the space of common experience is four-dimensional. It would therefore seem useful to demonstrate a specific higher dimensional theory (ten-dimensional Poincaré gauge theory) for which exact solutions of the vacuum field equations lead to a metric structure of a four-dimensional Minkowski space. Vacuum solutions with this property can then be viewed as an underlying state upon which matter fields may be erected that will retain a full ten-dimensional internal structure, but whose external asymptotic metric structure will remain four-dimensional.

1. DIRECT POINCARÉ GAUGE THEORY IN TEN DIMENSIONS

We start with a ten-dimensional Minkowski space M_{10} with a global coordinate cover $\{x^i \mid 0 \leq i \leq 9\}$, for which the line element assumes the

[1]Center for the Appliction of Mathematics, Lehigh University, Bethlehem, Pennsylvania 18015.

0020-7748/88/0500-0641$06.00/0

standard form

$$ds^2 = h_{ij}\, dx^i\, dx^j = (dx^0)^2 - \sum_{a=1}^{9} (dx^a)^2 \tag{1}$$

The Poincaré group for M_{10} is the 55-parameter Lie group $SO(1,9) \triangleright T(10)$, which we will denote by P_{55}. When this group is allowed to act locally, we obtain the 45 compensating 1-forms

$$W^\alpha = W_j^\alpha(x^m)\, dx^j, \qquad 1 \le \alpha \le 45 \tag{2}$$

for the $SO(1,9)$ sector, and the ten compensating 1-forms

$$\phi^i = \phi_j^i(x^m)\, dx^j, \qquad 0 \le i \le 9 \tag{3}$$

for the translation sector $T(10)$. Altogether, this gives a total of 550 Yang-Mills potential functions $\{W_j^\alpha, \phi_j^i \mid 0 \le i, j \le 9, 1 \le \alpha \le 45)$ for the local action of P_{55} on M_{10}.

The process for gauging P_{55} follows the identical pattern for gauging P_{10} that was described by Edelen (1985a-d); simply allow all Latin indices to range over 0-9 and all greek indices to range over 1-45. In particular, we obtain the ten distortion 1-forms

$$B^i = dx^i + W^\alpha l_{\alpha j}^{\ i} x^j + \phi^i, \qquad 0 \le i \le 9 \tag{4}$$

where the l's constitute a basis for the matrix Lie algebra of $SO(1,9)$. The B's serve to define the fundamental coframe fields and the line element on the resulting Riemann-Cartan space U_{10} by

$$dS^2 = B^i h_{ij} B^j = g_{ij}\, dx^i\, dx^j \tag{5}$$

The ten Cartan torsion 2-forms of U_{10} have the evaluation

$$\Sigma^i = DB^i = dB^i + W^\alpha l_{\alpha_j}^{\ i} \wedge B^j, \qquad 0 \le i \le 9 \tag{6}$$

While the 45 curvature 2-forms for the $SO(1,9)$ sector are given by

$$\theta^\alpha = dW^\alpha + \tfrac{1}{2} C_\beta{}^\alpha{}_\gamma W^\beta \wedge W^\gamma, \qquad 1 \le \alpha \le 45 \tag{7}$$

Here, $C_\beta{}^\alpha{}_\gamma$ are the structure constants for $so(1,9)$, and the corresponding components of the Cartan-Killing form on $so(1,9)$ are given by

$$C_{\alpha\beta} = C_\alpha{}^\rho{}_\gamma C_\beta{}^\gamma{}_\rho \tag{8}$$

Since $SO(1,9)$ is semisimple, the Cartan-Killing form on $so(1,9)$ is nonsingular.

In the interests of simplicity, we confine the discussion to the free gauge field problem for P_{55}. There will therefore be no matter field Lagrangian, and the same argument as that used in Edelen (1985a,d) indicates that we should also preclude the free P_{55} gauge field Lagrangian from depending on the components of the Cartan torsion 2-forms. We are therefore left with a free field Lagrangian that can only depend on the curvature 2-forms of the $SO(1,9)$ sector.

The free field Lagrangian must be invariant under the local action of P_{55}. Now, the simplest scalar invariant that can be formed from the curvature 2-forms is

$$U = -\tfrac{1}{4}\theta^{\alpha}_{ij}C_{\alpha\beta}\theta^{\beta}_{km}h^{ik}h^{jm} \tag{9}$$

Accordingly, a fairly standard free field Lagrangian is given by

$$L = KUB \tag{10}$$

where K is a coupling constant and

$$B = \det(B^i_j) \tag{11}$$

Let us set

$$S^k_i = \frac{\partial L}{\partial \phi^i_k} = \frac{\partial L}{\partial B^i_k} = KU\frac{\partial B}{\partial B^i_k} \tag{12}$$

$$H^{ij}_{\alpha} = \frac{\partial U}{\partial \theta^{\alpha}_{ij}} \tag{13}$$

so that

$$\frac{\partial L}{\partial \theta^{\alpha}_{ij}} = KBH^{ij}_{\alpha} \tag{14}$$

A direct analogy with the analysis given in Edelen (1985a) shows that the field equations for the free gauge fields for P_{55} are

$$S^k_i = KU\frac{\partial B}{\partial B^i_k} = 0 \tag{15}$$

for the translation-compensating fields, and

$$\partial_i(BH^{ij}_{\alpha}) - W^{\beta}_i C_{\beta}{}^{\gamma}{}_{\alpha}BH^{ij}_{\gamma} = 0 \tag{16}$$

for the compensating fields for the $SO(1,9)$ sector. An obvious rewriting of (16) gives us

$$B\{\partial_i H^{ij}_{\alpha} - W^{\beta}_i C_{\beta}{}^{\gamma}{}_{\alpha}H^{ij}_{\gamma}\} + H^{ij}_{\alpha}\partial_i B = 0 \tag{17}$$

2. DECOMPOSITION OF M_{10} AND THE FIELD EQUATIONS BY SUBGROUP SPLITTING OF P_{55}

The group P_{55} admits the subgroup inclusions

$$SO(1,9) \triangleright T(10) \supset SO(1,9) \supset SO(9) \supset SO(3)\times SO(3)\times SO(3) \tag{18}$$

The latter group $SO(3)\times SO(3)\times SO(3)$ engenders a natural decomposition of M_{10} in terms of isotropic three-dimensional spaces that are the domains of action of each of the $SO(3)$ factors. It is therefore useful to introduce new coordinate labels by

$$\{x^i \mid 0\le i\le 9\}=\{x^0, x_k^a=x^{3(k-1)+a} \mid 1\le a, k\le 3\} \tag{19}$$

Thus, $\{x_k^a \mid 1\le a\le 3\}$ are Cartesian coordinates on the kth isotropic three-dimensional Euclidean space on which the kth copy of $SO(3)$ acts.

It is natural to look for solutions of the field equations that are consistent with this decomposition of M_{10}. Since only $SO(3)\times SO(3)\times SO(3)$ is involved, a necessary requirement is that all of the compensating fields for the translation sector vanish:

$$\phi_j^i(x^m)=0, \qquad 0\le i, j\le 9 \tag{20}$$

Let us order the generators of $so(1,9)$ so that the first nine generate $so(3)\times so(3)\times so(3)$. We must then require that

$$W^\alpha=0, \qquad 10\le\alpha\le 45 \tag{21}$$

In view of the direct product structure that we are now dealing with, it is convenient to introduce an alternative designation for the first nine W's,

$$\{W^\alpha \mid 1\le\alpha\le 9\}=\{W^A[k]=W^{3(k-1)+A} \mid 1\le A, k\le 3\} \tag{22}$$

Thus, $W^A[1]$ are the compensating 1-forms for the local action of $SO(3)$ on the first three-dimensional space with coordinates x_1^a, $W^A[2]$ are compensating 1-forms for the local action of $SO(3)$ on the second three-dimensional space with coordinates x_2^a, etc.

This notation allows us to rewrite the distortion 1-forms as

$$\{B^i \mid 0\le i\le 9\}=\{B^0, B^a[k] \mid 1\le a, k\le 3\} \tag{23}$$

where

$$B^a[k]=dx_k^a+W^A[k]e_{Ab}{}^a x_k^b \tag{24}$$

and we have made use of the representation for the generating matrices of $so(3)$ in terms of the standard three-component permutation symbols. The line element on U_{10} thus becomes

$$dS^2=(dx^0)^2-\sum_{k=1}^{3} B^a[k]\delta_{ab}B^b[k] \tag{25}$$

In like manner, let us set

$$\{\theta^\alpha \mid 1\le\alpha\le 9\}=\{\theta^A[k]=\theta^{3(k-1)+A} \mid 1\le A, k\le 3\} \tag{26}$$

Since we are dealing with $so(3)\times so(3)\times so(3)$, and all of the W^{α} vanish for $\alpha>9$, the expressions for the surviving curvture 2-forms simplify significantly,

$$\theta^{A}[k]=dW^{a}[k]+\tfrac{1}{2}e^{A}{}_{BC}W^{B}[k]\wedge W^{C}[k] \tag{27}$$

We then have

$$4U=\sum_{k=1}^{3}\theta^{A}_{ij}[k]\delta_{AB}\theta^{B}_{mn}[k]h^{im}h^{jn} \tag{28}$$

and hence the nonzero field intensities are

$$H^{ij}_{A}[k]=\frac{\partial U}{\partial\theta^{A}_{ij}[k]} \tag{29}$$

When these evaluations are substituted into the field equations given in the previous section, there is also a drastic simplification. The only field equations not identically satisfied are

$$U\frac{\partial B}{\partial B^{i}_{k}}=0,\quad 0\le i,k\le 9 \tag{30}$$

and

$$B\{\partial_{i}H^{ij}_{A}[k]-W^{B}_{i}[k]e_{BA}{}^{C}H^{ij}_{C}\}+H^{ij}_{A}[k]\partial_{i}B=0 \tag{31}$$

$0\le j\le 9$; $1\le A, k\le 3$), where we have used the standard representation for the structure constants of $so(3)$ in terms of the three-component permutation symbols. We therefore have a total of 190 field equations for the determination of the 90 field variables $W^{A}_{i}[k](x^{m})$, and the problem is highly overdetermined at this level, where the nonzero W's are allowed to depend on all ten independent variables. We note, however, that all of these equations can be satisfied if the ten by ten matrix with entries $((B^{i}_{j}))$ has rank less than 9 (i.e., $B=0$, $\partial B/\partial B^{i}_{k}=0$).

3. TORSION-FREE SOLUTIONS OF THE FIELD EQUATIONS

We first note that $SO(3)$ and $SU(2)$ have the same Lie algebras, and hence solutions to the free field equations for local action of $SU(2)$ should prove to be useful in constructing solutions to the field equations just given. In fact, if the indices i and j are appropriately restricted in (31), the quantities within the curly brackets are exactly the field equations for free $SU(2)$ compensating fields. Now, there are many known $SU(2)$ solutions that could be used in various ways. The one we wish to concentrate on is the static "magnetic monopole" solution of Yang and Wu (1969). The restriction to the appropriate three-dimensional "factor" spaces of M_{10} that comes from the $so(3)\times so(3)\times so(3)$ subgroup of P_{55} can be achieved by setting

$$W^A[k]=dx_k^a\, e^A{}_{ab}x_k^b r_k^{-2} \tag{32}$$

that is, each $W^A[k]$ is a 1-form on the three-dimensional space with coordinates $\{x_k^a \mid 1\le a\le 3\}$ that define the Yang–Wu solution for $SU(2)$.

The first thing to be done is to substitute (32) into (24) in order to determine the 1-forms $B^a[k]$. As a prelude to this, we first obtain the evaluation

$$\rho^a[k]=W^A[k]e_{Ab}{}^a x_k^b=\frac{x_k^a}{r_k}\,dr_k-dx_k^a \tag{33}$$

since it will also be useful later. Thus, since equations (24) and (33) show that $B^a[k]=dx_k^a+\rho^a[k]$, we have the explicit evaluations

$$B^a[k]=\frac{x_k^a}{r_k}\,dr_k \tag{34}$$

Now, only one of each of the triplets of 1-forms $\{B^1[k],\ B^2[k],\ B^3[k]\}$ is independent, and hence the ten by ten matrix with entries B_j^i has rank 4,

$$\operatorname{rank}(B_j^i)=4 \tag{35}$$

We therefore have

$$B=\det(B_j^i)=0 \tag{36}$$

and each cofactor of the matrix $((B_j^i))$ also vanishes. This shows that

$$\partial B/\partial B_k^i=0 \tag{37}$$

and hence the field equations (30) and (31) are satisfied throughout M_{10}. In fact, B, its spatial derivatives, and the terms in the curly brackets in (31) each vanish throughout M_{10}, so one could say that we have "doubly solved" the field equations.

The next thing we establish is that the solution just obtained makes U_{10} torsion-free. In order to see this, we go back to the definition of the Cartan torsion:

$$\Sigma^i=DB^i=dB^i+W^\alpha l_{\alpha_j}^{\ i}\wedge B^j,\qquad 0\le i\le 9 \tag{38}$$

These decompose in the same way to yield $\Sigma^0=0$ and

$$\Sigma^a[k]=dB^a[k]+W^A[k]e_{Ab}{}^a\wedge B^b[k] \tag{39}$$

Use of equations (33) and (34) easily shows that

$$W^a[k]e_{Ab}{}^a\wedge B^b[k]=\rho^a[k]\wedge\frac{dr_k}{r_k}=-dx_k^a\wedge\frac{dr_k}{r_k} \tag{40}$$

while (33) shows that $dB^a[k]$ has the same evaluation with the opposite sign. Accordingly, (39) gives $\Sigma^a[k]=0$, and we have the desired result

$$\Sigma^i=0, \qquad 0\leq i\leq 9 \tag{41}$$

It is easily seen that the curvature 2-forms $\theta^A[k]$ do not vanish throughout U_{10}, and hence this ten-dimensional space is torsion-free with a nontrivial curvature structure. On the other hand, we have the alternative evaluation (Edelen, 1985a)

$$\Sigma^a[k]=\theta^A[k]e_{Ab}{}^a x_k^b=\theta_b^a[k]x_k^b=0 \tag{42}$$

Accordingly, the matrix of curvature 2-forms on each of the three three-dimensional isotropic subspaces has the corresponding three-dimensional radius vector in its kernel. Each three-dimensional isotropic subspace thus has vanishing radial curvature, so that the support of the curvature matrix is purely rotational. In fact, the rotational curvature is so great that all directions from the origin in each three-dimensional isotropic space can be identified relative to the metric geometry of the resulting space-time, as we shall see.

4. DIMENSION SHAVING AND THE RESULTING FOUR-DIMENSIONAL MINKOWSKI SPACE

The crucial information is provided by substituting the evaluations given by (34) into the line element for U_{10} in the form given by equation (25):

$$dS^2=(dx^0)^2-(dr_1)^2-(dr_2)^2-(dr_3)^2 \tag{43}$$

Since this is the line element of a four-dimensional Minkowski space with local coordinates $\{x^0, r_1, r_2, r_3\}$, the space U_{10} has been *shaved down* to a four-dimensional Minkowski space as far as its metric geometry is concerned. On the other hand, U_{10} has a nontrivial curvature structure, as we have already seen, but the support of this curvature is internal as far as the metric structure is concerned. Indeed, we may view (43) as the statement that only the radial separation in each of the three isotropic three-dimensional spaces of U_{10} contributes to the resulting metric structure, while the angular separations at the origins of each of the three isotropic three-dimensional subspaces of U_{10} are "rolled up" by the large, radially orthogonal matrices of curvature 2-forms. The curvature structures of U_{10} thus become curvature structures of "pinched off" internal projective spaces which do not leave footprints that can be detected by the resulting line element of the flat Minkowski space given by (43). There are several alternative interpretations of this result that readers can construct for themselves. Suffice it to say at this juncture that we have exhibited a ten-dimensional theory for which an exact solution of the free field equations with nontrivial curvatures serves to shave the metric structure down to that of a four-dimensional Minkowski space.

5. UNIVERSALITY OF THE DIMENSION-SHAVING SOLUTION

The way we stumbled on the dimension-shaving solution was by using the free field Lagrangian given by (9) and (10) and then noting that the quantities inside the curly brackets in (31) are the $SU(2)$ field equations. What makes the solution work, however, is the fact that the matrix with entries B^i_j has rank 4. It thus turns out that surviving field equations (30) and (31) are satisfied no matter what invariant scalar is used for U, because we have $B=0$ and $\partial B/\partial B^i_k=0$. The dimension-shaving solution is therefore universal with respect to the choice of the free field, P_{55}-invariant Lagrangian that is independent of the components of the Cartan torsion. We therefore have dimension having for any free field Lagrangian of the form

$$L=KV(\theta^{\alpha}_{ij})B \tag{44}$$

where V is an arbitrary, P_{55}-invariant, scalar-valued function of its indicated arguments. In particular, we can have

$$V=K_0+K_1\theta^{\alpha}_{ij}l^{\,j}_{\alpha_k}h^{ki}+K_2U+\cdots \tag{45}$$

where the K's are coupling constants. The first two terms in (45) give the P_{55} analog of the Einstein–Hilbert Lagrangian, while U is the P_{55} analog of the Lagrangian for electromagnetism that is so often used in gauge theory.

Universality of the dimension-shaving, free field solution is of particular importance in the larger context in which matter field Lagrangians are included. The analysis of such problems can proceed by allowing the free field Lagrangian to have the general form given by (44), and then determining the shape of the function V in order to model specific structures. Once V has been determined in this way, soliton-like solutions for the matter fields should lead to asymptotically free gauge fields, and these in turn will contain remnants of the universal dimension-shaving solution provided the $SO(3)\times SO(3)\times SO(3)$ symmetry is not broken. There is therefore a reasonable prospect of obtaining ten-dimensional descriptions of matter that induce an asymptotic four-dimensional metric structure.

REFERENCES

Edelen, D. G. B. (1985a). *International Journal of Theoretical Physics*, **24**, 659.

Edelen, D. G. B. (1985b). *International Journal of Theoretical Physics*, **24**, 1091.

Edelen, D. G. B. (1985c). *International Journal of Theoretical Physics*, **24**, 1133.

Edelen, D. G. B. (1985d). *International Journal of Theoretical Physics*, **24**, 1173.

Yang, C. N., and Wu, T. T. (1969). *Properties of Matter Under Unusual Conditions*, H. Mark and S. Fernbach, eds., Interscience, New York.